International Journal of Theoretical Physics, Vol. 37, No. 10, 1998

Chaotic Maps, Control Parameter, and Liapunov Exponent

W.-H. Steeb,[1] M. A. van Wyk,[1] and R. Stoop[1]

Received February 27, 1998

We describe a one-dimensional chaotic map where the Liapunov exponent is a smooth function of a control parameter.

It is commonly believed that in chaotic systems the Liapunov exponent is a discontinuous function of a control parameter (Steeb, 1994). Examples are the Hénon map or driven anharmonic oscillators. Here we give a one-dimensional map f_a: $[0,1] \to [0, 1]$, where a is a control parameter, and show that the Liapunov exponent is a smooth function of the control parameter a. The map is given by

$$f_a(x) := \begin{cases} \dfrac{1-a}{a} x & \text{for } x \in [0, a) \\ \dfrac{2a}{1-2a} x + \dfrac{1-3a}{1-2a} & \text{for } x \in [a, 1/2) \\ \dfrac{2a}{1-2a} (1-x) + \dfrac{1-3a}{1-2a} & \text{for } x \in [1/2, 1-a] \\ \dfrac{1-a}{a} (1-x) & \text{for } x \in [1-a, 1] \end{cases}$$

where $a \in (0,1/2)$. The map is continuous, but not differentiable at the points a, $1 - a$ ($a \neq 1/3$), and $x = 1/2$. The map is piecewise linear. For $a = 1/3$ we obtain the tent map. The tent map is ergodic and chaotic (Steeb, 1996).

[1] International School for Scientific Computing, Rand Afrikaans University, Auckland Park 2006, South Africa; e-mail: WHS@NA.RAU.AC.ZA

The Liapuonov exponent for almost all initial conditions is given by ln(2) and the invariant density is given by $\rho(x) = 1$. The map f_a is a special bungalow-tent map. The intersection point P of the line in the interval $[1/2, 1 - a)$ and the line in the interval $[1 - a, 1]$ lies on the diagonal $y = x$.

We apply the Frobenius–Perron integral equation (Steeb, 1994, 1996; Van Wyk and Steeb, 1997; Steeb *et al.*, 1994) to find the Liapunov exponent. The Frobenius–Perron integral equation is given by

$$\rho_a(x) = \int_0^1 \rho_a(y)\delta(x - f_a(y))\, dy$$

We apply the identities for the delta function

$$\delta(cy) \equiv \frac{1}{|c|}\,\delta(y)$$

and

$$\delta(g(y)) \equiv \sum_n \frac{1}{|g'(y_n)|}\,\delta(y - y_n)$$

where the sum runs over all zeros with multiplicity 1 and $g'(y_n)$ denotes the derivative of g taken at y_n. Taking these identities into account and differentiation in the sense of generalized functions (Steeb, 1996), we obtain the invariant density

$$\rho_a(x) = \frac{1}{2 - 3a}\,\chi_{[0,1-a]}(x) + \frac{1 - 2a}{a(2 - 3a)}\,\chi_{(1-a,1]}(x)$$

where χ is the indicator function, i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Thus the invariant density is constant in the interval $[0,1 - a)$. At $1 - a$ the invariant density jumps to another constant value. In our calculations we have to consider two domains for x, $[0,1 - a)$ and $[1 - a, 1]$. The Liapunov exponent is calculated using

$$\lambda(a) = \int_0^1 \rho_a(x)\,\ln\left|\frac{df_a}{dx}\right|\, dx$$

where we used differentiation in the sense of generalized functions. Thus we find that the Liapunov exponent as a smooth function of the control paramter a is given by

$$\lambda(a) = \frac{1 - a}{2 - 3a}\ln\left(\frac{1 - a}{a}\right) + \frac{1 - 2a}{2 - 3a}\ln\left(\frac{2a}{1 - 2a}\right)$$

For $a = 1/3$ we obviously obtain $\lambda(1/3) = \ln 2$. This is the Liapunov exponent for the tent map. For $a \to 0$ we obtain $\lambda(a \to 0) = \frac{1}{2} \ln 2$. For $a \to 1/2$ we obtain $\lambda(a \to 1/2) = 0$. The $\lambda(a)$ has a maximum for $a = 1/3$ (tent map). Furthermore $\lambda(a)$ is a convex function in the interval (0, 1/2).

Finally we mention that the problem can also be solved using a Markov partition and the thermodynamic formalism (Stoop and Steeb, 1997).

REFERENCES

Steeb, W.-H. (1994). *Chaos und Quantenchaos in Dynamischen Systemen*, Spektrum Akademischer Verlag, Heidelberg.

Steeb, W.-H. (1996). *Problems and Solutions in Theoretical and Mathematical Physics*, Vol. II, World Scientific, Singapore.

Steeb, W.-H., Solms, F., and Stoop R. (1994). *Journal of Physics A,* **27**, L399.

Stoop, R., and Steeb, W.-H. (1997). *Physical Review E,* **55**, 7763.

Van Wyk, M. A., and Steeb, W.-H. (1997). *Chaos in Electronics*, Kluwer, Dordrecht.